Blowing up solutions for an elliptic Neumann problem with sub- and supercritical nonlinearity. Part I: N=3

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Blowing up Solutions for an Elliptic Neumann Problem with Sub- or Supercritical Nonlinearity Part I: $N = 3$

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Abstract

We consider the sub- or supercritical Neumann elliptic problem $-\Delta u + \mu u = u^{5+\varepsilon}$, $u > 0$ in Ω; $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, Ω being a smooth bounded domain in $\mathbb{R}^3$, $\mu > 0$ and $\varepsilon \neq 0$ a small number. H_μ denoting the regular part of the Green's function of the operator $-\Delta + \mu$ in Ω with Neumann boundary conditions, and $\varphi_\mu(x) = \mu^{\frac{1}{2}} + H_\mu(x, x)$, we show that a nontrivial relative homology between the level sets φ_μ^c and φ_μ^b, $b < c < 0$, induces the existence, for $\varepsilon > 0$ small enough, of a solution to the problem, which blows up as ε goes to zero at a point $a \in \Omega$ such that $b \leq \varphi_\mu(a) \leq c$. The same result holds, for $\varepsilon < 0$, assuming that $0 < b < c$. It is shown that, $M_\mu = \sup_{x\in\Omega} \varphi_\mu(x) < 0$ (resp. > 0) for μ small (resp. large) enough, providing us with cases where the above assumptions are satisfied.

1 Introduction

In this paper we consider the nonlinear Neumann elliptic problem

$$(P_{q,\mu}) \qquad \begin{cases} -\Delta u + \mu u &= u^q \quad u > 0 & \text{in } \Omega \\ \quad\ \frac{\partial u}{\partial n} &= 0 & \text{on } \partial\Omega \end{cases}$$

where $1 < q < +\infty$, $\mu > 0$ and Ω is a smooth and bounded domain in $\mathbb{R}^3$.

Equation $(P_{q,\mu})$ arises in many branches of the applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer-Meinhardt system in biological pattern formation ([14], [26]) or of parabolic equations in chemotaxis, e.g. Keller-Segel model ([24]).

When q is subcritical, i.e. $q < 5$, Lin, Ni and Takagi proved that the only solution, for small μ, is the constant one, whereas nonconstant solutions appear for large μ [24] which blow up, as μ goes to infinity, at one or several points. The least energy solution blows up at a boundary

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point which maximizes the mean curvature of the frontier [28][29]. Higher energy solutions exist which blow up at one or several points, located on the boundary [8][13][22][33][19], in the interior of the domain [5][7][11][12][16][21][38][40], or some of them on the boundary and others in the interior [18]. (A good review can be found in [26].) In the critical case, i.e. $q = 5$, Zhu [41] proved that, for convex domains, the only solution is the constant one for small μ (see also [39]). For large μ, nonconstant solutions exist [1][34]. As in the subcritical case the least energy solution blows up, as μ goes to infinity, at a unique point which maximizes the mean curvature of the boundary [3][27]. Higher energy solutions have also been exhibited, blowing up at one [2][35][31][15] or several boundary points [25][36][37][17]. The question of interior blow-up is still open. However, in contrast with the subcritical situation, at least one blow-up point has to lie on the boundary [32]. Very few is known about the supercritical case, save the uniqueness of the radial solution on a ball for small μ [23].

Our aim, in this paper, is to study the problem for fixed μ, when the exponent q close to the critical one, i.e. $q = 5 + \varepsilon$ and ε is a small nonzero number. Whereas the previous results, concerned with peaked solutions, always assume that μ goes to infinity, we are going to prove that a single peak solution may exist for finite μ, provided that q is close enough to the critical exponent. Such a solution blows up, as q goes to 5, at one point which may be characterized.

In order to state a precise result, some notations have to be introduced. Let $G_\mu(x, y)$ denote the Green's function of the operator $-\Delta + \mu$ in Ω with Neumann boundary conditions. Namely, for any $y \in \Omega$, $x \mapsto G_\mu(x, y)$ is the unique solution of

$$-\Delta G_\mu + \mu G_\mu = 4\pi\delta_y \quad x \in \Omega\,; \quad \frac{\partial G_\mu}{\partial n} = 0 \quad x \in \partial\Omega. \tag{1.1}$$

G_μ writes as

$$G_\mu(x, y) = \frac{e^{-\mu^{1/2}|x-y|}}{|x-y|} - H_\mu(x, y)$$

where $H_\mu(x, y)$, regular part of the Green's function, satisfies

$$-\Delta H_\mu + \mu H_\mu = O \quad x \in \Omega\,; \quad \frac{\partial H_\mu}{\partial n} = \frac{1}{\partial n}\left(\frac{e^{-\mu^{1/2}|x-y|}}{|x-y|}\right) \quad x \in \partial\Omega. \tag{1.2}$$

We set

$$\varphi_\mu(x) = \mu^{\frac{1}{2}} + H_\mu(x, x).$$

It is to be noticed that

$$H_\mu(x, x) \to -\infty \quad \text{as} \quad d(x, \partial\Omega) \to 0 \tag{1.3}$$

implying that

$$M_\mu = \sup_{x\in\Omega} \varphi_\mu(x)$$

is achieved in Ω. (See (5.10) in Proposition 5.2 for the proof of (1.3).) Denoting

$$f^\alpha = \{x \in \Omega, f(x) \leq \alpha\}$$

the level sets of a function f defined in Ω, we have :

Theorem 1.1 *Assume that there exist b and c, $b < c < 0$, such that c is not a critical value of φ_μ and the relative homology $H_*(\varphi_\mu^c, \varphi_\mu^b) \neq 0$. $(P_{5+\varepsilon,\mu})$ has a nontrivial solution, for $\varepsilon > 0$ close enough to zero, which blows up as ε goes to zero at a point $a \in \Omega$, such that $b < \varphi_\mu(a) < c$.*
The same result holds, for $\varepsilon < 0$, assuming that $0 < b < c$.

We notice that, $M_\mu < 0$ (resp. > 0) when μ is small (resp. large) enough (see (5.12) and (5.13) of Proposition 5.2). Consequently, we deduce from the previous result :

Theorem 1.2 *There exist μ_0 and μ_1, $0 < \mu_0 \leq \mu_1$, such that :*
1) If $0 < \mu < \mu_0$, $(P_{5+\varepsilon,\mu})$ has a nontrivial solution, for $\varepsilon > 0$ close enough to zero, which blows up as ε goes to zero at a maximum point a of $H_\mu(a,a)$.
2) If $\mu > \mu_1$, $(P_{5+\varepsilon,\mu})$ has a nontrivial solution, for $\varepsilon < 0$ close enough to zero, which blows up as ε goes to zero at a maximum point a of $H_\mu(a,a)$.

Remarks. 1) In the critical case, i.e. $\varepsilon = 0$, further computations suggest that a nontrivial solution should exist for $\mu > \mu_0$ close enough to μ_0, such that $M_\mu > 0$ and $M_{\mu_0} = 0$. This solution would blow up, as previously, at a maximum point of $H_{\mu_0}(a,a)$ as μ goes to μ_0. (This contrasts to previous results for $P_{5,0}$ on the nonexistence of solutions for μ small ([39], [41]) and nonexistence of interior bubble solutions for μ large ([10],[31]).)

2) In a forthcoming paper, we shall treat the case $N \geq 4$, which appears to be qualitatively different.

The scheme of the proof is the following. In the next section, we define a two-parameters set of approximate solutions to the problem, and we look for a true solution in a neighborhood of this set. Considering in Section 3 the linearized problem at an approximate solution, and inverting it in suitable functional spaces, the problem reduces to a finite dimensional one, which is solved in Section 4. Some useful facts and computations are collected in Appendix.

2 Approximate solutions and rescaling

For sake of simplicity, we consider in the following the supercritical case, i.e. we assume that $\varepsilon > 0$. The subcritical case may be treated exactly in the same way.

For normalization reasons, we consider throughout the paper the equation

$$-\Delta u + \mu u = 3u^{5+\varepsilon}, \quad u > 0 \tag{2.1}$$

instead of the original one. The solutions are identical, up to the multiplicative constant $3^{-\frac{1}{4+\varepsilon}}$. We recall that, according to [6], the functions

$$U_{\lambda,a}(x) = \frac{\lambda^{\frac{1}{2}}}{(1+\lambda^2|x-a|^2)^{\frac{1}{2}}} \qquad \lambda > 0\,,\, a \in \mathbb{R}^3 \tag{2.2}$$

are the only solutions to the problem

$$-\Delta u = 3u^5, \quad u > 0 \qquad \text{in} \quad \mathbb{R}^3.$$

As $a \in \Omega$ and λ goes to infinity, these functions provide us with approximate solutions to the problem that we are interested in. However, in view of the additional linear term μu which occurs in $(P_{5+\varepsilon,\mu})$, the approximation needs to be improved. Actually, we define in Ω the following functions :

$$\tilde{U}_{\lambda,a,\mu}(x) = U_{\lambda,a}(x) - \frac{1}{\lambda^{\frac{1}{2}}}\Big(\frac{1-e^{-\mu^{\frac{1}{2}}|x-a|}}{|x-a|} + H_\mu(a,x)\Big)$$

which satisfy

$$-\Delta\tilde{U}_{\lambda,a,\mu} + \mu\tilde{U}_{\lambda,a,\mu} = 3U^5_{\lambda,a} + \mu\Big(U_{\lambda,a} - \frac{1}{\lambda^{\frac{1}{2}}|x-a|}\Big). \tag{2.3}$$

We are going to seek for solutions in a neighborhood of such functions, with the a priori assumption that a remains far from the boundary of the domain, that is there exists some number $\delta > 0$ such that

$$d(a,\partial\Omega) > \delta. \tag{2.4}$$

Moreover, integral estimates (see Appendix) suggest to make the additional *a priori* assumption that λ behaves as $1/\varepsilon$ as ε goes to zero. Namely, we set

$$\lambda = \frac{1}{\Lambda\varepsilon} \qquad\qquad \frac{1}{\delta'} < \Lambda < \delta' \tag{2.5}$$

with δ' some strictly positive number.

In fact, in order to avoid the singularity which appears in the right hand side of (2.3), and to cancel the normal derivative on the boundary, we modify slightly the definition of $\tilde{U}_{\lambda,a,\mu}$, setting

$$V_{\Lambda,a,\mu,\varepsilon}(x) = \tilde{U}_{\frac{1}{\Lambda\varepsilon},a,\mu}(x) - \frac{\mu}{2}(\Lambda\varepsilon)^{\frac{1}{2}}|x-a|(1-e^{-\frac{\varepsilon^2}{|x-a|^2}}) + \theta_{\Lambda,a,\mu,\varepsilon}(x) \tag{2.6}$$

$\theta_{\Lambda,a,\mu,\varepsilon} = \theta$ being the unique solution to the problem

$$\begin{cases} -\Delta\theta + \mu\theta &= 0 \hspace{14em} \text{in } \ \Omega \\ \quad\frac{\partial\theta}{\partial n} &= \frac{\partial}{\partial n}\Big(-U_{\frac{1}{\Lambda\varepsilon},a}(x) + \frac{(\Lambda\varepsilon)^{\frac{1}{2}}}{|x-a|} + \frac{\mu}{2}(\Lambda\varepsilon)^{\frac{1}{2}}|x-a|(1-e^{-\frac{\varepsilon^2}{|x-a|^2}})\Big) \quad \text{on } \ \partial\Omega. \end{cases}$$

From the above assumption (2.4) we know that

$$H_\mu(a,x) = O(1) \qquad\qquad \theta_{\lambda,a,\mu,\varepsilon} = O(\varepsilon^{\frac{5}{2}}) \tag{2.7}$$

in $C^2(\Omega)$. We note that $V_{\Lambda,a,\mu,\varepsilon} = V$ satisfies

$$\begin{cases} -\Delta V + \mu V &= 3U^5_{\frac{1}{\Lambda\varepsilon},a} + \mu\Big(U_{\frac{1}{\Lambda\varepsilon},a} - \frac{(\Lambda\varepsilon)^{\frac{1}{2}}}{|x-a|}e^{-\frac{\varepsilon^2}{|x-a|^2}}\Big) \quad -\frac{\mu\Lambda^{\frac{1}{2}}\varepsilon^{\frac{5}{2}}}{|x-a|^3}\Big(1+\frac{2\varepsilon^2}{|x-a|^2}\Big)e^{-\frac{\varepsilon^2}{|x-a|^2}} \\ & \qquad -\frac{\mu^2\varepsilon^2}{2}(\Lambda\varepsilon)^{\frac{1}{2}}|x-a|(1-e^{-\frac{\varepsilon^2}{|x-a|^2}}) \quad \text{in } \ \Omega \\ \quad\frac{\partial V}{\partial n} &= 0 \hspace{16em} \text{on } \ \partial\Omega. \end{cases} \tag{2.8}$$

The $V_{\Lambda,a,\mu,\varepsilon}$'s are the suitable approximate solutions in the neigbourhood of which we shall find a true solution to the problem. In order to make further computations easier, we proceed to a rescaling. We set

$$\Omega_\varepsilon = \frac{\Omega}{\varepsilon}$$

and we define in Ω_ε the functions

$$W_{\Lambda,\xi,\mu,\varepsilon}(x) = \varepsilon^{\frac{1}{2}} V_{\Lambda,a,\mu,\varepsilon}(\varepsilon x) \qquad\qquad \xi = \frac{a}{\varepsilon} \tag{2.9}$$

which write as

$$\begin{aligned} W_{\Lambda,\xi,\mu,\varepsilon}(x) = U_{\frac{1}{\Lambda},\xi}(x) - \Lambda^{\frac{1}{2}}\Big(\frac{1-e^{-\mu^{\frac{1}{2}}\varepsilon|x-\xi|}}{|x-\xi|} + H_{\mu,\varepsilon}(\xi,x)\Big) \\ - \frac{\mu\varepsilon^2}{2}\Lambda^{\frac{1}{2}}|x-\xi|(1-e^{-\frac{1}{|x-\xi|^2}}) + \tilde{\theta}_{\Lambda,\xi,\mu,\varepsilon}(x) \end{aligned} \tag{2.10}$$

$H_{\mu,\varepsilon}$ denoting the regular part of the Green's function of the operator $-\Delta+\mu\varepsilon^2$ with Neumann boundary conditions in Ω_ε, and $\tilde{\theta}_{\Lambda,\xi,\mu,\varepsilon}(x) = \varepsilon^{\frac{1}{2}}\theta_{\Lambda,a,\mu,\varepsilon}(\varepsilon x)$. We notice that, taking account of (2.7)

$$H_{\mu,\varepsilon}(\xi,x) = O(\varepsilon) \qquad\qquad \tilde{\theta}_{\Lambda,\xi,\mu,\varepsilon}(x) = O(\varepsilon^3) \tag{2.11}$$

in $C^2(\Omega_\varepsilon)$. We notice also that assumption (2.4) is equivalent to

$$d(\xi,\partial\Omega_\varepsilon) > \frac{\delta}{\varepsilon} \tag{2.12}$$

and that $W_{\Lambda,\xi,\mu,\varepsilon} = W$ satisfies the uniform estimate $|W_{\Lambda,\xi,\mu,\varepsilon}| \le CU_{\frac{1}{\Lambda},\xi}$ in Ω_ε. Moreover, we have

$$\begin{cases} -\Delta W + \mu\varepsilon^2 W & = 3U^5_{\frac{1}{\Lambda},\xi} + \mu\varepsilon^2\Big(U_{\frac{1}{\Lambda},a} - \frac{\Lambda^{\frac{1}{2}}}{|x-\xi|}e^{-\frac{1}{|x-\xi|^2}}\Big) \quad -\frac{\mu\Lambda^{\frac{1}{2}}\varepsilon^2}{|x-\xi|^3}\Big(1+\frac{2}{|x-\xi|^2}\Big)e^{-\frac{1}{|x-\xi|^2}} \\ & \quad -\frac{\mu^2\varepsilon^4}{2}(\Lambda\varepsilon)^{\frac{1}{2}}|x-\xi|(1-e^{-\frac{1}{|x-\xi|^2}}) \qquad \text{in } \Omega_\varepsilon \\ \frac{\partial W}{\partial n} & = 0 \qquad\qquad \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2.13}$$

Finding a solution to $(P_{5+\varepsilon,\mu})$ in a neighbourhood of the functions $V_{\Lambda,a,\mu,\varepsilon}$ is equivalent, through the rescaling, to solving the problem

$$(P'_{5+\varepsilon,\mu}) \qquad \begin{cases} -\Delta u + \mu\varepsilon^2 u & = 3u^{5+\varepsilon} \quad u > 0 \qquad \text{in } \Omega_\varepsilon \\ \frac{\partial u}{\partial n} & = 0 \qquad\qquad\qquad \text{on } \partial\Omega_\varepsilon \end{cases} \tag{2.14}$$

in a neigbourhood of the functions $W_{\Lambda,\xi,\mu,\varepsilon}$. For that purpose, we have to use some local inversion procedure. Namely, we are going to look for a solution to $(P'_{\varepsilon,\mu})$ writing as

$$w = W_{\Lambda,\xi,\mu,\varepsilon} + \omega$$

with ω small and orthogonal at $W_{\Lambda,\xi,\mu,\varepsilon}$, in a suitable sense, to the manifold

$$M = \Big\{W_{\Lambda,\xi,\mu,\varepsilon},\ (\Lambda,\xi) \text{ satisfying (2.5) (2.12)}\Big\}. \tag{2.15}$$

The general strategy consists in finding first, using an inversion procedure, a smooth map $(\Lambda,\xi) \mapsto \omega(\Lambda,\xi)$ such that $W_{\Lambda,\xi,\mu,\varepsilon} + \omega(\Lambda,\xi,\mu,\varepsilon)$ solves the problem in an orthogonal space to M. Then, we are left with a finite dimensional problem, for which a solution may be found using the topological assumption of the theorem. In the subcritical or critical case, the first step may be performed in H^1 (see e.g. [4][30][31]). However, this approach is not valid any more in the supercritical case, for H^1 does not inject into L^q as $q > 6$. Following [9], we use instead weighted Hölder spaces to reduce the problem to a finite dimensional one.

3 The finite dimensional reduction

3.1 Inversion of the linearized problem

We first consider the linearized problem at a function $W_{\Lambda,\xi,\mu,\varepsilon}$, and we invert it in an orthogonal space to M. From now on, we omit for sake of simplicity the indices in the writing of $W_{\Lambda,\xi,\mu,\varepsilon}$. Equipping $H^1(\Omega_\varepsilon)$ with the scalar product

$$(u,v)_\varepsilon = \int_{\Omega_\varepsilon} (\nabla u \cdot \nabla v + \mu\varepsilon^2 uv)$$

orthogonality to the functions

$$Y_0 = \frac{\partial W}{\partial \Lambda} \qquad Y_i = \frac{\partial W}{\partial \xi_i} \qquad 1 \le i \le 3 \tag{3.1}$$

in that space is equivalent, setting

$$Z_0 = -\Delta \frac{\partial W}{\partial \Lambda} + \mu\varepsilon^2 \frac{\partial W}{\partial \Lambda} \qquad Z_i = -\Delta \frac{\partial W}{\partial \xi_i} + \mu\varepsilon^2 \frac{\partial W}{\partial \xi_i} \qquad 1 \le i \le 3 \tag{3.2}$$

to the orthogonality in $L^2(\Omega_\varepsilon)$, equipped with the usual scalar product $\langle \cdot,\cdot \rangle$, to the functions Z_i, $0 \le i \le 3$. Then, we consider the following problem : $h \in L^\infty(\Omega_\varepsilon)$ being given, find a function ϕ which satisfies

$$\left\{ \begin{array}{rll} -\Delta\phi + \mu\varepsilon^2\phi - 3(5+\varepsilon)W_+^{4+\varepsilon}\phi & = h + \sum_i c_i Z_i & \text{in} \quad \Omega_\varepsilon \\ \frac{\partial \phi}{\partial n} & = 0 & \text{on} \quad \partial\Omega_\varepsilon \\ < Z_i, \phi > & = 0 & 0 \le i \le 3 \end{array} \right. \tag{3.3}$$

for some numbers c_i.

Existence and uniqueness of ϕ will follow from an inversion procedure in suitable functional spaces. Namely, for f a function in Ω_ε, we define the following weighted L^∞-norms

$$\|f\|_* = \sup_{x \in \Omega_\varepsilon} \left| (1 + |x-\xi|^2)^{\frac{1}{2}} f(x) \right|$$

and

$$\|f\|_{**} = \sup_{x \in \Omega_\varepsilon} \left| (1 + |x-\xi|^2)^2 f(x) \right|.$$

Writing U instead of $U_{\frac{1}{\Lambda},\xi}$, the first norm is equivalent to $\|U^{-1}f\|_\infty$ and the second one to $\|U^{-4}f\|_\infty$, uniformly with respect to ξ and Λ.

We have the following result :

Proposition 3.1 *There exists $\varepsilon_0 > 0$ and a constant $C > 0$, independent of ε and ξ, Λ satisfying (2.12) (2.5), such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in L^\infty(\Omega_\varepsilon)$, problem* (3.3) *has a unique solution $\phi \equiv L_\varepsilon(h)$. Besides,*

$$\|L_\varepsilon(h)\|_* \le C\|h\|_{**} \qquad |c_i| \le C\|h\|_{**}. \tag{3.4}$$

Moreover, the map $L_\varepsilon(h)$ is C^2 with respect to Λ, ξ and the L^∞_*-norm, and

$$\|D_{(\Lambda,\xi)}\, L_\varepsilon(h)\|_* \le C\|h\|_{**} \qquad \|D^2_{(\Lambda,\xi)}\, L_\varepsilon(h)\|_* \le C\|h\|_{**}. \tag{3.5}$$

Proof. The argument follows closely the ideas in [9]. We repeat it for convenience of the reader. The proof relies on the following result :

Lemma 3.1 *Assume that ϕ_ε solves (3.3) for $h = h_\varepsilon$. If $\|h_\varepsilon\|_{**}$ goes to zero as ε goes to zero, so does $\|\phi_\varepsilon\|_*$.*

Proof of the lemma. For $0 < \rho < 1$, we define

$$\|f\|_\rho = \sup_{x\in\Omega_\varepsilon} |(1 + |x-\xi|^2)^{\frac{1}{2}(1-\rho)} f(x)|$$

and we first prove that $\|\phi_\varepsilon\|_\rho$ goes to zero. Arguing by contradiction, we may assume that $\|\phi_\varepsilon\|_\rho = 1$. Multiplying the first equation in (3.3) by Y_j and integrating in Ω_ε we find

$$\sum_i c_i\langle Z_i, Y_j\rangle = \langle -\Delta Y_j + \mu\varepsilon^2 Y_j - 3(5+\varepsilon)W_+^{4+\varepsilon}Y_j, \phi_\varepsilon\rangle - \langle h_\varepsilon, Y_j\rangle.$$

On one hand we check, in view of the definition of Z_i, Y_j

$$\langle Z_0, Y_0\rangle = \|Y_0\|_\varepsilon^2 = \gamma_0 + o(1) \qquad \langle Z_i, Y_i\rangle = \|Y_i\|_\varepsilon^2 = \gamma_1 + o(1) \qquad 1 \le i \le 3 \tag{3.6}$$

where γ_0, γ_1 are strictly positive constants, and

$$\langle Z_i, Y_j\rangle = o(1) \qquad i \ne j. \tag{3.7}$$

On the other hand, in view of the definition of Y_j and W, straightforward computations yield

$$\langle -\Delta Y_j + \mu\varepsilon^2 Y_j - 3(5+\varepsilon)W_+^{4+\varepsilon}Y_j, \phi_\varepsilon\rangle = o(\|\phi_\varepsilon\|_\rho)$$

and

$$\langle h_\varepsilon, Y_j\rangle = O(\|h_\varepsilon\|_{**}).$$

Consequently, inverting the quasi diagonal linear system solved by the c_i's, we find

$$c_i = O(\|h_\varepsilon\|_{**}) + o(\|\phi_\varepsilon\|_\rho). \tag{3.8}$$

In particular, $c_i = o(1)$ as ε goes to zero. The first equation in (3.3) may be written as

$$\phi_\varepsilon(x) = 3(5+\varepsilon)\int_{\Omega_\varepsilon} G_\varepsilon(x,y)\Big(W_+^{4+\varepsilon}\phi_\varepsilon + h_\varepsilon + \sum_i c_i Z_i\Big)dy \tag{3.9}$$

for all $x \in \Omega_\varepsilon$, G_ε denoting the Green's function of the operator $(-\Delta + \mu\varepsilon^2)$ in Ω_ε with Neumann boundary conditions.

We notice that by scaling and (5.11) of Proposition 5.2,

$$G_\varepsilon(x,y) = \varepsilon G_\mu(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}) \le \frac{C}{|x-y|} \tag{3.10}$$

and hence we obtain

$$
\begin{aligned}
\left|\int_{\Omega_\varepsilon} G_\varepsilon(x,y)W_+^{4+\varepsilon}\phi_\varepsilon dy\right| &\le C\|\phi_\varepsilon\|_\rho \int_{\Omega_\varepsilon}\frac{1}{|x-y|}\frac{1}{(1+|x-\xi|^2)^{\frac{1}{2}(3+\varepsilon+\rho)}}dy\\
&\le C\|\phi_\varepsilon\|_\rho(1+|x-\xi|^2)^{-\frac{1}{2}}\\
\left|\int_{\Omega_\varepsilon} G_\varepsilon(x,y)h_\varepsilon dy\right| &\le C\|h_\varepsilon\|_{**}\int_{\Omega_\varepsilon}\frac{1}{|x-y|}\frac{1}{(1+|x-\xi|^2)^2}dy\\
&\le C\|h_\varepsilon\|_{**}(1+|x-\xi|^2)^{-\frac{1}{2}}\\
\left|\int_{\Omega_\varepsilon} G_\varepsilon(x,y)Z_i dy\right| &\le C\int_{\Omega_\varepsilon}\frac{1}{|x-y|}\frac{1}{(1+|x-\xi|^2)^{\frac{5}{2}}}dy\\
&\le C(1+|x-\xi|^2)^{-\frac{1}{2}}
\end{aligned}
\tag{3.11}
$$

from which we deduce

$$(1+|x-\xi|^2)^{\frac{1}{2}(1-\rho)}|\phi_\varepsilon(x)| \le C(1+|x-\xi|^2)^{-\frac{\rho}{2}}.$$

$\|\phi_\varepsilon\|_\rho = 1$ implies the existence of $R>0$, $\gamma>0$ independent of ε such that $\|\phi_\varepsilon\|_{L^\infty(B_R(\xi))} > \gamma$. Then, elliptic theory shows that along some subsequence, $\tilde{\phi}_\varepsilon(x) = \phi_\varepsilon(x-\xi)$ converges uniformly in any compact subset of $\mathbb{R}^3$ to a nontrivial solution of

$$-\Delta\tilde{\phi} = 15U^4_{\tilde{\Lambda},0}\tilde{\phi}$$

for some $\tilde{\Lambda}>0$. Moreover, $|\tilde{\phi}(x)| \le C/|x|$. As a consequence, $\tilde{\phi}$ writes as

$$\tilde{\phi} = \alpha_0\frac{\partial U_{\tilde{\Lambda},0}}{\partial\tilde{\Lambda}} + \sum_{i=1}^{3}\alpha_i\frac{\partial U_{\tilde{\Lambda},0}}{\partial a_i}$$

(see e.g. [30]). On the other hand, equalities $<Z_i,\phi_\varepsilon>=0$ provide us with the equalities

$$
\begin{aligned}
&\int_{\mathbb{R}^3} -\Delta\frac{\partial U_{\tilde{\Lambda},0}}{\partial\tilde{\Lambda}}\tilde{\phi} = \int_{\mathbb{R}^3} U^4_{\tilde{\Lambda},0}\frac{\partial U_{\tilde{\Lambda},0}}{\partial\tilde{\Lambda}}\tilde{\phi} = 0\\
&\int_{\mathbb{R}^3} -\Delta\frac{\partial U_{\tilde{\Lambda},0}}{\partial a_i}\tilde{\phi} = \int_{\mathbb{R}^3} U^4_{\tilde{\Lambda},0}\frac{\partial U_{\tilde{\Lambda},0}}{\partial a_i}\tilde{\phi} = 0 \qquad 1\le i\le 3.
\end{aligned}
$$

As we have also

$$\int_{\mathbb{R}^3}|\nabla\frac{\partial U_{\tilde{\Lambda},0}}{\partial\tilde{\Lambda}}|^2 = \gamma_0 > 0 \qquad\qquad \int_{\mathbb{R}^3}|\nabla\frac{\partial U_{\tilde{\Lambda},0}}{\partial a_i}|^2 = \gamma_1 > 0 \quad 1\le i\le 3$$

and

$$\int_{\mathbb{R}^3}\nabla\frac{\partial U_{\tilde{\Lambda},0}}{\partial\tilde{\Lambda}}.\nabla\frac{\partial U_{\tilde{\Lambda},0}}{\partial a_i} = \int_{\mathbb{R}^3}\nabla\frac{\partial U_{\tilde{\Lambda},0}}{\partial a_j}.\nabla\frac{\partial U_{\tilde{\Lambda},0}}{\partial a_i} = 0 \quad i\ne j$$

the α_j's solve a homogeneous quasi diagonal linear system, yielding $\alpha_j = 0$, $0\le\alpha_j\le 3$, and $\tilde{\phi}=0$, hence a contradiction. This proves that $\|\phi_\varepsilon\|_\rho = o(1)$ as ε goes to zero. Furthermore, (3.9), (3.11) and (3.8) show that

$$\|\phi_\varepsilon\|_* \le C(\|h_\varepsilon\|_{**} + \|\phi_\varepsilon\|_\rho)$$

whence also $\|\phi_\varepsilon\|_* = o(1)$ as ε goes to zero.

Proof of Proposition 3.1 completed. We set

$$H = \Big\{\phi \in H^1(\Omega_\varepsilon),\ < Z_i, \phi >= 0 \quad 0 \le i \le 3\Big\}$$

equipped with the scalar product $(\cdot,\cdot)_\varepsilon$. Problem (3.3) is equivalent to finding $\phi \in H$ such that

$$(\phi,\theta)_\varepsilon = \langle 3(5+\varepsilon)W_+^{4+\varepsilon}\phi + h\,,\,\theta\rangle \qquad \forall \theta \in H$$

that is

$$\phi = T_\varepsilon(\phi) + \tilde{h} \tag{3.12}$$

$\tilde{h}$ depending linearly on h, and T_ε being a compact operator in H. Fredholm's alternative ensures the existence of a unique solution, provided that the kernel of $Id - T_\varepsilon$ is reduced to 0. We notice that $\phi_\varepsilon \in Ker(Id - T_\varepsilon)$ solves (3.3) with $h = 0$. Thus, we deduce from Lemma 3.1 that $\|\phi_\varepsilon\|_* = o(1)$ as ε goes to zero. As $Ker(Id - T_\varepsilon)$ is a vector space, $Ker(Id - T_\varepsilon) = \{0\}$. The inequalities (3.4) follow from Lemma 3.1 and (3.8). This completes the proof of the first part of Proposition 3.1.

The smoothness of L_ε with respect to Λ and ξ is a consequence of the smoothness of T_ε and $\tilde{h}$, which occur in the implicit definition (3.12) of $\phi \equiv L_\varepsilon(h)$, with respect to these variables. Inequalities (3.5) are obtained differentiating (3.3), writing the derivatives of ϕ with respect Λ and ξ as a linear combination of the Z_i' and an orthogonal part, and estimating each term using the first part of the proposition - see [9] [20] for detailed computations. □

3.2 The reduction

In view of (2.13), a first correction between the approximate solution W and a true solution to $(P'_{\varepsilon,\mu})$ writes as

$$\psi^\varepsilon = L_\varepsilon(R^\varepsilon) \tag{3.13}$$

with

$$\begin{aligned} R^\varepsilon =& 3W_+^{5+\varepsilon} - (-\Delta W + \mu\varepsilon^2 W) \\ =& 3W_+^{5+\varepsilon} - 3U^5_{\frac{1}{\Lambda},\xi} - \mu\varepsilon^2\Big(U_{\frac{1}{\Lambda},a} - \frac{\Lambda^{\frac12}}{|x-\xi|}e^{-\frac{1}{|x-\xi|^2}}\Big) + \frac{\mu\Lambda^{\frac12}\varepsilon^2}{|x-\xi|^3}\Big(1+\frac{1}{|x-\xi|^2}\Big)e^{-\frac{1}{|x-\xi|^2}} \\ &+ \frac{\mu^2\varepsilon^4}{2}(\Lambda\varepsilon)^{\frac12}|x-\xi|(1-e^{-\frac{1}{|x-\xi|^2}}). \end{aligned} \tag{3.14}$$

We have :

Lemma 3.2 *There exists C, independent of ξ, Λ satisfying (2.12) (2.5), such that*

$$\|R^\varepsilon\|_{**} \le C\varepsilon \qquad \|D_{(\Lambda,\xi)}R^\varepsilon\|_{**} \le C\varepsilon \qquad \|D^2_{(\Lambda,\xi)}R^\varepsilon\|_{**} \le C\varepsilon.$$

Proof. According to (2.10), $W = U + O(\varepsilon)$ uniformly in Ω_ε. Consequently, noticing that $U \ge C\varepsilon$ in Ω_ε, C independent of ε

$$U^5 - W_+^{5+\varepsilon} = O(\varepsilon U^5|\ln U| + \varepsilon U^4)$$

uniformly in Ω_ε, whence

$$\|U^5 - W_+^{5+\varepsilon}\|_{**} \le C\|U^{-4}(U^5 - W_+^{5+\varepsilon})\|_\infty = O(\varepsilon).$$

On the other hand

$$(1+|x-\xi|^2)^2\left[\mu\varepsilon^2\Big(U_{\frac{1}{\Lambda},a} - \frac{\Lambda^{\frac12}}{|x-\xi|}e^{-\frac{1}{|x-\xi|^2}}\Big) - \frac{\mu\Lambda^{\frac12}\varepsilon^2}{|x-\xi|^3}\Big(1+\frac{1}{|x-\xi|^2}\Big)e^{-\frac{1}{|x-\xi|^2}}\right.$$
$$\left. - \frac{\mu^2\varepsilon^4}{2}(\Lambda\varepsilon)^{\frac12}|x-\xi|(1-e^{-\frac{1}{|x-\xi|^2}})\right] = O(\varepsilon)$$

uniformly for $x \in \Omega_\varepsilon$, since

$$U_{\frac{1}{\Lambda},a} - \frac{\Lambda^{\frac12}}{|x-\xi|}e^{-\frac{1}{|x-\xi|^2}} = O(|x-\xi|^{-3})$$

as $|x-\xi|$ goes to infinity, and $|x-\xi| = O(1/\varepsilon)$ in Ω_ε. The first estimate of the lemma follows. The others are obtained in the same way, differentiating (3.14) and estimating each term as previously. □

Lemma 3.2 and Proposition 3.1 yield :

Lemma 3.3 *There exists C, independent of ξ, Λ satisfying (2.12) (2.5), such that*

$$\|\psi^\varepsilon\|_* \le C\varepsilon \qquad \|D_{(\Lambda,\xi)}\psi^\varepsilon\|_* \le C\varepsilon \qquad \|D^2_{(\Lambda,\xi)}\psi^\varepsilon\|_* \le C\varepsilon.$$

We consider now the following nonlinear problem : finding ϕ such that, for some numbers c_i

$$\begin{cases} -\Delta(W+\psi+\phi) + \mu\varepsilon^2(W+\psi+\phi) - 3(W+\psi+\phi)_+^{5+\varepsilon} = \sum_i c_i Z_i & \text{in} \quad \Omega_\varepsilon \\ \frac{\partial\phi}{\partial n} = 0 & \text{on} \quad \partial\Omega_\varepsilon \\ < Z_i, \phi > = 0 & 0 \le i \le 3. \end{cases} \tag{3.15}$$

Setting

$$N_\varepsilon(\eta) = (W+\eta)_+^{5+\varepsilon} - W_+^{5+\varepsilon} - (5+\varepsilon)W_+^{4+\varepsilon}\eta \tag{3.16}$$

the first equation in (3.15) writes as

$$-\Delta\phi + \mu\varepsilon^2\phi - 3(5+\varepsilon)W_+^{4+\varepsilon}\phi = 3N_\varepsilon(\psi+\phi) + \sum_i c_i Z_i \tag{3.17}$$

for some numbers c_i. Assuming that $\|\eta\|_*$ is bounded, say $\|\eta\|_* \le M$ for some constant M, we have

$$\|N_\varepsilon(\eta)\|_{**} \le C\|\eta\|_*^2$$

whence, assuming that $\|\phi\|_* \le 1$ and using Lemma 3.3

$$\|N_\varepsilon(\psi+\phi)\|_{**} \le C(\|\phi\|_*^2 + \varepsilon^2). \tag{3.18}$$

We state the following result :

Proposition 3.2 *There exists C, independent of ε and ξ, Λ satisfying (2.12) (2.5), such that for small ε problem (3.15) has a unique solution $\phi = \phi(\Lambda, \xi, \mu, \varepsilon)$ with*

$$\|\phi\|_* \leq C\varepsilon^2. \tag{3.19}$$

Moreover, $(\Lambda, \xi) \mapsto \phi(\Lambda, \xi, \mu, \varepsilon)$ is C^2 with respect to the L^∞_*-norm, and

$$\|D_{(\Lambda,\xi)}\phi\|_* \leq C\varepsilon^2 \qquad \|D^2_{(\Lambda,\xi)}\phi\|_* \leq C\varepsilon^2. \tag{3.20}$$

Proof. Following [9], we consider the map A_ε from $\mathcal{F} = \{\phi \in H^1 \cap L^\infty(\Omega_\varepsilon) \,:\, \|\phi\|_* \leq \varepsilon\}$ to $H^1 \cap L^\infty(\Omega_\varepsilon)$ defined as

$$A_\varepsilon(\phi) = L_\varepsilon(3N_\varepsilon(\phi + \psi))$$

and we remark that finding a solution ϕ to problem (3.15) is equivalent to finding a fixed point of A_ε. One the one hand we have, for $\phi \in \mathcal{F}$

$$\|A_\varepsilon(\phi)\|_* \leq \|L_\varepsilon(N_\varepsilon(\phi + \psi))\|_* \leq \|N_\varepsilon(\phi + \psi)\|_{**} \leq C\varepsilon^2 \leq \varepsilon$$

for ε small enough, implying that A_ε sends $\mathcal{F}$ into itself. On the other hand A_ε is a contraction. Indeed, for ϕ_1 and ϕ_2 in $\mathcal{F}$, we write

$$\begin{aligned}\|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* &\leq \|N_\varepsilon(\psi + \phi_1) - N_\varepsilon(\psi + \phi_2)\|_{**} \\ &\leq \|U^{-4}\big(N_\varepsilon(\psi + \phi_1) - N_\varepsilon(\psi + \phi_2)\big)\|_\infty.\end{aligned}$$

In view of (3.16) we have

$$\partial_\eta N_\varepsilon(\eta) = (5 + \varepsilon)\big((W + \eta)_+^{4+\varepsilon} - W_+^{4+\varepsilon}\big) \tag{3.21}$$

whence

$$|N_\varepsilon(\psi + \phi_1) - N_\varepsilon(\psi + \phi_2)| \leq CU^3|\psi + t\phi_1 + (1 - t)\phi_2||\phi_1 - \phi_2|$$

for some $t \in (0, 1)$. Then

$$\begin{aligned}\|U^{-4}\big(N_\varepsilon(\psi + \phi_1) - N_\varepsilon(\psi + \phi_2)\big)\|_\infty &\leq C\|U^{-1}(\psi + t\phi_1 + (1 - t)\phi_2)(\phi_1 - \phi_2)\|_\infty \\ &\leq C(\|\psi\|_* + \|\phi_1\|_* + \|\phi_2\|_*)\|\phi_1 - \phi_2\|_* \\ &\leq \varepsilon\|\phi_1 - \phi_2\|_*.\end{aligned}$$

This implies that A_ε has a unique fixed point in $\mathcal{F}$, that is problem (3.15) has a unique solution ϕ such that $\|\phi\|_* \leq \varepsilon$. Furthermore, the definition of ϕ as a fixed point of A_ε yields

$$\|\phi\|_* = \|L_\varepsilon(N_\varepsilon(\phi + \psi))\|_* \leq C\|N_\varepsilon(\phi + \psi)\|_{**} \leq C\varepsilon^2$$

using (3.18), whence (3.19).

In order to prove that $(\Lambda, \xi) \mapsto \phi(\Lambda, \xi)$ is C^2, we remark that setting for $\eta \in \mathcal{F}$

$$B(\Lambda, \xi, \eta) \equiv \eta - L_\varepsilon(3N_\varepsilon(\eta + \psi))$$

ϕ is defined as

$$B(\Lambda, \xi, \phi) = 0. \tag{3.22}$$

We have

$$\partial_\eta B(\Lambda, \xi, \eta)[\theta] = \theta - 3L_\varepsilon\big(\theta\,(\partial_\eta N_\varepsilon)(\eta + \psi)\big)$$

and, using (3.21)

$$\begin{aligned}\|L_\varepsilon\big(\theta\,(\partial_\eta N_\varepsilon)(\eta+\psi)\big)\|_* &\le C\|\theta\,(\partial_\eta N_\varepsilon)(\eta+\psi)\|_{**}\\ &\le C\|U^{-3}(\partial_\eta N_\varepsilon)(\eta+\psi)\|_\infty\|\theta\|_*\\ &\le C\|\eta+\psi\|_*\|\theta\|_*\\ &\le C\varepsilon\|\theta\|_*.\end{aligned}$$

Consequently, $\partial_\eta B(\Lambda,\xi,\phi)$ is invertible in L^∞_* with uniformly bounded inverse. Then, the fact that $(\Lambda,\xi)\mapsto\phi(\Lambda,\xi)$ is C^2 follows from the fact that $(\Lambda,\xi,\eta)\mapsto L_\varepsilon(N_\varepsilon(\eta+\psi))$ is C^2 and the implicit functions theorem.

Finally, let us show how estimates (3.20) may be obtained. Derivating (3.22) with respect to Λ, we have

$$\partial_\Lambda\phi = 3(\partial_\eta B(\Lambda,\xi,\phi))^{-1}\Big((\partial_\Lambda L_\varepsilon)(N_\varepsilon(\phi+\psi))+L_\varepsilon((\partial_\Lambda N_\varepsilon)(\phi+\psi))+L_\varepsilon((\partial_\eta N_\varepsilon)(\phi+\psi)\partial_\Lambda\psi)\Big)$$

whence, according to Proposition 3.1

$$\|\partial_\Lambda\phi\|_*\le C\Big(\|N_\varepsilon(\phi+\psi)\|_{**}+\|(\partial_\Lambda N_\varepsilon)(\phi+\psi)\|_{**}+\|(\partial_\eta N_\varepsilon)(\phi+\psi)\partial_\Lambda\psi\|_{**}\Big).$$

From (3.18) and (3.19) we know that

$$\|N_\varepsilon(\phi+\psi)\|_{**}\le C\varepsilon^2.$$

Concerning the next term, we notice that according to the definition (3.16) of N_ε

$$\begin{aligned}|(\partial_\Lambda N_\varepsilon)(\phi+\psi)| &= (5+\varepsilon)\Big|(W+\phi+\psi)_+^{4+\varepsilon}-W_+^{4+\varepsilon}-(4+\varepsilon)W_+^{3+\varepsilon}(\phi+\psi)\Big||\partial_\Lambda W|\\ &\le CU^5\|\phi+\psi\|_*^2\\ &\le CU^5\varepsilon^2\end{aligned}$$

using again (3.18) and (3.19), whence

$$\|(\partial_\Lambda N_\varepsilon)(\phi+\psi)\|_{**}\le C\varepsilon^2.$$

Lastly, from (3.21) we deduce

$$|(\partial_\eta N_\varepsilon)(\phi+\psi)\partial_\Lambda\psi|\le U^5\|\phi+\psi\|_*\|\partial_\Lambda\psi\|_*$$

yielding

$$\|(\partial_\eta N_\varepsilon)(\phi+\psi)\partial_\Lambda\psi\|_{**}\le C\varepsilon^2.$$

Finally we obtain

$$\|\partial_\Lambda\phi\|_*\le C\varepsilon^2.$$

The other first and second derivatives of ϕ with respect to Λ and ξ may be estimated in the same way (see [20] for detailed computations concerning the second derivatives). This concludes the proof of Proposition 3.2. $\square$

3.3 Coming back to the original problem

We introduce the following functional defined in $H^1(\Omega) \cap L^{6+\varepsilon}(\Omega)$

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \mu u^2) - \frac{3}{6+\varepsilon}\int_\Omega u_+^{6+\varepsilon} \tag{3.23}$$

whose nontrivial critical points are solutions to $(P_{5+\varepsilon,\mu})$ (up to the multiplicative constant $3^{\frac{1}{4+\varepsilon}}$). We consider also the rescaled functions defined in Ω

$$\hat{W}(\Lambda, a)(x) = \varepsilon^{-\zeta} W_{\Lambda,\xi}(\varepsilon^{-1}x) = \varepsilon^{\frac{1}{2}-\zeta} V_{\Lambda,a}(x) \tag{3.24}$$

with

$$\zeta = \frac{1}{2+\frac{1}{2}\varepsilon} \qquad\qquad a = \varepsilon\xi.$$

We define also

$$\hat{\psi}(\Lambda, a)(x) = \varepsilon^{-\zeta}\psi(\Lambda,\xi)(\varepsilon^{-1}x) \qquad\qquad \hat{\phi}(\Lambda, a)(x) = \varepsilon^{-\zeta}\phi(\Lambda,\xi)(\varepsilon^{-1}x) \tag{3.25}$$

and we set

$$I_\varepsilon(\Lambda, a) \equiv J_\varepsilon\big((\hat{W} + \hat{\psi} + \hat{\phi})(\Lambda, a)\big). \tag{3.26}$$

We have :

Proposition 3.3 *The function $u = 3^{\frac{1}{4+\varepsilon}}(\hat{W} + \hat{\psi} + \hat{\phi})$ is a solution to problem $(P_{5+\varepsilon,\mu})$ if and only if (Λ, a) is a critical point of I_ε.*

Proof. For v in $H^1(\Omega_\varepsilon) \cap L^{6+\varepsilon}(\Omega_\varepsilon)$, we set

$$K_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} (|\nabla v|^2 + \mu\varepsilon^2 v^2) - \frac{3}{6+\varepsilon}\int_{\Omega_\varepsilon} v_+^{6+\varepsilon} \tag{3.27}$$

whose nontrivial critical points are solutions to $(P'_{5+\varepsilon,\mu})$. According to the definition I_ε we have

$$I_\varepsilon(\Lambda, a) = \varepsilon^{1-2\zeta} K_\varepsilon\big((W + \psi + \phi)(\Lambda, \xi)\big). \tag{3.28}$$

We notice that $u = 3^{\frac{1}{4+\varepsilon}}(\hat{U}+\hat{\psi}+\hat{\phi})$ being a solution to $(P_{5+\varepsilon,\mu})$ is equivalent to $W+\psi+\phi$ being a solution to $(P'_{5+\varepsilon,\mu})$, that is a critical point of K_ε. It is also equivalent to the cancellation of the c_i's in (3.15) or, in view of (3.6) (3.7)

$$K'_\varepsilon(W + \psi + \phi)[Y_i] = 0 \qquad 0 \le i \le 3. \tag{3.29}$$

On the other hand, we deduce from (3.28) that $I'_\varepsilon(\Lambda, a) = 0$ is equivalent to the cancellation of $K'_\varepsilon(W + \psi + p)$ applied to the derivatives of $W + \psi + p$ with respect to Λ and ξ. According to the definition (3.1) of the Y_i's, Lemma 3.3 and Proposition 3.2 we have

$$\frac{\partial(W+\psi+\phi)}{\partial\Lambda} = Y_0 + y_0 \qquad\qquad \frac{\partial(W+\psi+\phi)}{\partial\xi_j} = Y_j + y_j \qquad 1 \le j \le 3$$

with $\|y_i\|_{L^\infty_*} = o(1)$, $0 \le i \le 3$. Writing

$$y_i = y'_i + \sum_{j=0}^{3} a_{ij} Y_j \qquad\qquad \langle y'_i, Z_j\rangle = (y'_i, Y_j)_\varepsilon = 0 \qquad 0 \le i, j \le 3$$

and

$$K'_\varepsilon(W+\psi+p)[Y_i]=\alpha_i$$

it turns out that $I'_\varepsilon(\Lambda,a)=0$ is equivalent, since $K'_\varepsilon(W+\psi+p)[\theta]=0$ for $\langle\theta,Z_j\rangle=(\theta,Y_j)_\varepsilon=0$, $0\le j\le 3$, to

$$(Id+[a_{ij}])[\alpha_i]=0.$$

As $a_{ij}=O(\|y_i\|_*)=o(1)$, we see that $I'_\varepsilon(\Lambda,a)=0$ means exactly that (3.29) is satisfied. □

4 Proof of Theorem 1.1

In view of Proposition 3.3 we have, for proving the theorem, to find critical points of I_ε. We establish first a C^2-expansion of I_ε.

4.1 Expansion of I_ε

Proposition 4.1 *There exist A, B, C, strictly positive constants such that*

$$I_\varepsilon(\Lambda,a)=A+\frac{A}{4}\varepsilon\ln(\varepsilon\Lambda)+\frac{1}{2}(C+\frac{A}{6})\varepsilon+\frac{3B\Lambda}{2}\big(\mu^{1/2}+H_\mu(a,a)\big)\varepsilon+\varepsilon\sigma_\varepsilon(\Lambda,a)$$

with σ_ε, $D_{(\Lambda,a)}\sigma_\varepsilon$ and $D^2_{(\Lambda,a)}\sigma_\varepsilon$ going to zero as ε goes to zero, uniformly with respect to a, Λ satisfying (2.4) and (2.5).

Proof. In view of the definition (3.26) of I_ε, we first estimate $J_\varepsilon(\hat W)$. We have

$$\begin{aligned}\varepsilon^{2\zeta-1}J_\varepsilon(\hat W)&=\varepsilon^{2\zeta-1}J_\varepsilon(\varepsilon^{\frac12-\zeta}V)\\&=J_\varepsilon(V)+3\frac{1-\varepsilon^{\frac{\varepsilon}{2}}}{6+\varepsilon}\int_\Omega V_+^{6+\varepsilon}\\&=J_\varepsilon(V)+\frac12\Big(-\frac{\varepsilon}{2}\ln\varepsilon+o(\varepsilon)\Big)\int_\Omega V_+^{6+\varepsilon}\end{aligned}$$

from which we deduce, using the integral estimates (5.8), (5.9) and Proposition 5.1 in Appendix,that

$$J_\varepsilon(\hat W)=A+\frac{A}{4}\varepsilon\ln(\varepsilon\Lambda)+\frac12(C+\frac{A}{6})\varepsilon+\frac{3B\Lambda}{2}(\mu^{1/2}+H_\mu(a,a))\varepsilon+o(\varepsilon).\tag{4.1}$$

Then, we prove that

$$I_\varepsilon(\Lambda,a)-J_\varepsilon(\hat W+\hat\psi)=o(\varepsilon).\tag{4.2}$$

Indeed, from a Taylor expansion and the fact that $J'_\varepsilon(\hat W+\hat\psi+\hat\phi)[\phi]=0$, we have

$$\begin{aligned}I(\Lambda,a)-J_\varepsilon(\hat W+\hat\psi)&=J_\varepsilon(\hat W+\hat\psi+\hat\phi)-J_\varepsilon(\hat W+\hat\psi)\\&=\int_0^1 J''_\varepsilon(\hat W+\hat\psi+t\hat\phi)[\hat\phi,\hat\phi]tdt\\&=\varepsilon^{1-2\zeta}\int_0^1 K''_\varepsilon(W+\psi+\phi)[\phi,\phi]tdt\\&=\varepsilon^{1-2\zeta}\int_0^1\Big(\int_{\Omega_\varepsilon}\big(|\phi|^2+\mu\varepsilon^2\phi^2-3(5+\varepsilon)(W+\psi+\phi)_+^{4+\varepsilon}\phi^2\big)\Big)tdt\\&=\varepsilon^{1-2\zeta}\int_0^1\Big(\int_{\Omega_\varepsilon}\big(N_\varepsilon(\phi+\psi)\phi+3(5+\varepsilon)\big[W_+^{4+\varepsilon}-(W+\psi+t\phi)_+^{4+\varepsilon}\big]\phi^2\big)\Big)tdt.\end{aligned}$$

The desired result follows from (3.18), Lemma 3.3 and (3.19). Similar computations show that estimate (4.2) is also valid for the first and second derivatives of $I_\varepsilon(\Lambda, a) - J_\varepsilon(\hat{W} + \hat{\psi})$ with respect to Λ and a. Then, the proposition will follow from an estimate of $J_\varepsilon(\hat{W} + \hat{\psi}) - J_\varepsilon(\hat{W})$. We have

$$\begin{aligned} J_\varepsilon(\hat{W} + \hat{\psi}) - J_\varepsilon(\hat{W}) =& \varepsilon^{1-2\zeta}\big(K_\varepsilon(W + \psi) - K_\varepsilon(W)\big) \\ =& \varepsilon^{1-2\zeta}\big(K_\varepsilon^{'}(W)[\psi] + \int_0^1 (1-t)K_\varepsilon^{''}(W + t\psi)[\psi, \psi]\big). \end{aligned}$$

By definition of ψ and R^ε

$$K_\varepsilon^{'}(W)[\psi] = -\int_{\Omega_\varepsilon} R^\varepsilon \psi$$

and we have

$$K_\varepsilon^{''}(W + t\psi)[\psi, \psi] = \int_{\Omega_\varepsilon} (|\nabla \psi|^2 + \mu\varepsilon^2\psi^2) - 3(5+\varepsilon)\int_{\Omega_\varepsilon} (W + t\psi)_+^{4+\varepsilon}\psi^2.$$

Then, integration by parts and $\psi = L_\varepsilon(R^\varepsilon)$ yield

$$K_\varepsilon^{''}(W + t\psi)[\psi, \psi] = \int_{\Omega_\varepsilon} R^\varepsilon \psi - 3(5+\varepsilon)\int_{\Omega_\varepsilon} \big((W + t\psi)_+^{4+\varepsilon} - W_+^{4+\varepsilon}\big)\psi^2.$$

Consequently

$$\begin{aligned} & J_\varepsilon(\hat{W} + \hat{\psi}) - J_\varepsilon(\hat{W}) \\ & = \varepsilon^{1-2\zeta}\left(-\frac{1}{2}\int_{\Omega_\varepsilon} R^\varepsilon\psi - 3(5+\varepsilon)\int_0^1 (1-t)(\int_{\Omega_\varepsilon} \big[(W+t\psi)_+^{4+\varepsilon} - W_+^{4+\varepsilon}\big]\psi^2)dt\right) \end{aligned}$$

and Lemmas 3.2 and 3.3 yield

$$J_\varepsilon(\hat{W} + \hat{\psi}) - J_\varepsilon(\hat{W}) = o(\varepsilon).$$

The same estimate holds for the first and second derivatives with respect to Λ and a, obtained similarly with more delicate computations - see Proposition 3.4 in [20]. This concludes the proof of Proposition 4.1. □

4.2 Proof of Theorem 1.1 completed

According to the statement of Theorem 1.1, we assume the existence of b and c, $b < c < 0$, such that c is not a critical value of $\varphi_\mu(x) = \mu^{\frac{1}{2}} + H_\mu(x, x)$ and the relative homology $H_*(\varphi_\mu^c, \varphi_\mu^b) \neq 0$. In view of Proposition 3.3, we have to prove the existence of a critical point of $I_\varepsilon(\Lambda, a)$. According to Proposition 4.1, we have

$$\frac{\partial I_\varepsilon}{\partial \Lambda}(\Lambda, a) = \frac{A\varepsilon}{4\Lambda} + \frac{3B}{2}\varphi_\mu(a)\varepsilon + o(\varepsilon)$$

and

$$\frac{\partial^2 I_\varepsilon}{\partial \Lambda^2}(\Lambda, a) = -\frac{A\varepsilon}{4\Lambda^2} + o(\varepsilon)$$

uniformly with respect to a and Λ satisfying (2.4) (2.5). For $\delta > 0$, $\eta > 0$, we define

$$\Omega_{\delta,\gamma} = \Big\{ a \in \Omega \text{ s.t. } d(a, \partial\Omega) > \delta,\ \varphi_\mu(a) < -\gamma \Big\}.$$

The implicit functions theorem provides us, for ε small enough, with a C^1-map $a \in \Omega_{\delta,\gamma} \mapsto \Lambda(a)$ such that

$$\frac{\partial I_\varepsilon}{\partial \Lambda}(\Lambda(a), a) = 0 \qquad \Lambda(a) = -\frac{A}{6B}(\varphi_\mu(a))^{-1} + o(1).$$

Then, finding a critical point of $(\Lambda, a) \mapsto I_\varepsilon(\Lambda, a)$ reduces to finding a critical point of $a \mapsto \tilde{I}_\varepsilon(a)$, with

$$\tilde{I}_\varepsilon(a) = I_\varepsilon(\Lambda(a), a).$$

We deduce from Proposition 4.1 yields the C^1-expansion

$$\tilde{I}_\varepsilon(a) = A + \frac{A}{4}\varepsilon \ln \varepsilon + \frac{1}{2}\Big(C - \frac{A}{3} + \frac{A}{2} \ln \frac{A}{6B}\Big)\varepsilon - \frac{A}{4}\varepsilon \ln |\varphi_\mu(a)| + o(\varepsilon).$$

Therefore, up to an additive and to a multiplicative constant, we have to look for critical points in $\Omega_{\delta,\gamma}$ of

$$\mathcal{I}_\varepsilon(a) = -\ln |\varphi_\mu(a)| + \tau_\varepsilon(a) \tag{4.3}$$

with $\tau_\varepsilon(a) = o(1)$, $\nabla\tau_\varepsilon(a) = o(1)$ as ε goes to zero, uniformly with respect to $a \in \Omega_{\delta,\gamma}$.

Arguing by contradiction, we assume

(H) $\quad \mathcal{I}_\varepsilon$ has no critical point $a \in \Omega_{\delta,\gamma}$ such that $b < \varphi_\mu(a) < c$.

We are going to use the gradient of $\mathcal{I}_\varepsilon$ to build a continuous deformation of φ_μ^c onto φ_μ^b, a contradiction with the assumption $H_*(\varphi_\mu^c, \varphi_\mu^b) \neq 0$.

We first remark that φ_μ has isolated critical values, since φ_μ is analytic in Ω and $\varphi_\mu = -\infty$ on the boundary of Ω. Therefore, the assumption that c is not a critical value of φ_μ implies the existence of $\eta > 0$ such that φ_μ has no critical value in $(b, b+\eta] \cup (c-\eta, c]$. Moreover, φ_μ^c retracts by deformation onto $\varphi_\mu^{c-\eta}$, $\varphi_\mu^{b+\eta}$ retracts by deformation onto φ_μ^b, and $H_*(\varphi_\mu^{c-\eta}, \varphi_\mu^{b+\eta}) \neq 0$.

Secondly, we choose $\delta > 0$ such that $\varphi_\mu(x) < b$ for $d(x, \partial\Omega) \leq \delta$. We choose also $\gamma > 0$ such that $-\gamma > c$. Then, a point x in the complementary of $\Omega_{\delta,\gamma}$ in Ω is either in φ_μ^b, or not in φ_μ^c. Consequently, deforming $\varphi_\mu^{c-\eta}$ onto $\varphi_\mu^{b+\eta}$ is equivalent to deforming $\varphi_\mu^{c-\eta} \cap \Omega_{\delta,\gamma}$ onto $\varphi_\mu^{b+\eta}$. To this end we set, for $a_0 \in (\varphi_\mu^{c-\eta} \cap \Omega_{\delta,\gamma})$

$$\frac{d}{dt}a(t) = -\nabla \mathcal{I}_\varepsilon(a(t)) \qquad a(0) = a_0.$$

$a(t)$ is defined as long as the boundary of $\Omega_{\delta,\gamma}$ is not achieved. $\mathcal{I}_\varepsilon(a(t))$ being decreasing, (4.3) shows that for ε small enough, $a(t)$ remains in φ_μ^c. Then, the boundary of $\Omega_{\delta,\gamma}$ may only be achieved by $a(t)$ in φ_μ^b. This means that $a(t)$ is well defined as long as $b < \varphi_\mu(a(t)) < c$, and according to assumption (H), $\mathcal{I}_\varepsilon(a(t))$ is strictly decreasing in that region. Therefore (4.3) proves, for ε small enough, the existence of $t > 0$ such that $\varphi_\mu(a(t)) = b + \eta$. Composing the flow with a retraction of φ_μ^c onto $\varphi_\mu^{c-\eta}$, we obtain a continuous deformation of $\varphi_\mu^{c-\eta}$ onto $\varphi_\mu^{b+\eta}$, a contradiction with $H_*(\varphi_\mu^{c-\eta}, \varphi_\mu^{b+\eta}) \neq 0$.

The previous arguments prove the existence, for ε small enough, of a nontrivial solution u_ε to the problem

$$-\Delta u + \mu u = u_+^{5+\varepsilon} \text{ in } \Omega\,; \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

Then, the strong maximum principle shows that $u_\varepsilon > 0$ in Ω. The fact that u_ε blows up, as ε goes to zero, at a point a such that $b < \varphi_\mu(a) < c$, $\nabla\varphi_\mu(a) = 0$, follows from the construction of u_ε. In particular, $\nabla\varphi_\mu(a) = 0$ is a straightforward consequence of (4.3) as ε goes to zero. This concludes the proof of the theorem.

5 Appendix

5.1 Integral estimates

In this subsection, we collect the integral estimates which are needed in the previous section. We recall that according to the definitions of Section 2, we have

$$V_{\Lambda,a,\mu,\varepsilon}(x) = U_{\frac{1}{\Lambda\varepsilon},a}(x) - (\Lambda\varepsilon)^{\frac{1}{2}}\Big(\frac{1-e^{-\mu^{\frac{1}{2}}|x-a|}}{|x-a|} + H_\mu(a,x)\Big) + \rho_{\Lambda,a,\mu,\varepsilon}(x) \tag{5.1}$$

with

$$\rho_{\Lambda,a,\mu,\varepsilon} = O(|\varepsilon|^{\frac{3}{2}}) \tag{5.2}$$

uniformly in Ω and with respect to a and Λ satisfying (2.4) (2.5), and the same estimate holds for the derivatives of $\rho_{\Lambda,a,\mu,\varepsilon}$ with respect to a and Λ. We recall also that $V_{\Lambda,a,\mu,\varepsilon}$ satisfies

$$\begin{cases} -\Delta V_{\Lambda,a,\mu,\varepsilon} + \mu V_{\Lambda,a,\mu,\varepsilon} & = 3U^5_{\frac{1}{\Lambda\varepsilon},a} + \mu\left(U_{\frac{1}{\Lambda\varepsilon},a} - \frac{(\Lambda\varepsilon)^{\frac{1}{2}}}{|x-a|}\right) + \rho'_{\Lambda,a,\mu,\varepsilon} \quad \text{in } \Omega \\ \frac{\partial V_{\Lambda,a,\mu,\varepsilon}}{\partial n} & = 0 \quad \text{on } \partial\Omega \end{cases} \tag{5.3}$$

with

$$\rho'_{\Lambda,a,\mu,\varepsilon} = \mu\frac{(\Lambda\varepsilon)^{\frac{1}{2}}}{|x-a|}(1-e^{-\frac{\varepsilon^2}{|x-a|^2}}) - \mu(\Lambda\varepsilon)^{\frac{1}{2}}\left(\frac{\varepsilon^2}{|x-a|^3} + \frac{2\varepsilon^4}{|x-a|^5}\right)e^{-\frac{\varepsilon^2}{|x-a|^2}} + O(|\varepsilon|^{\frac{7}{2}}) \tag{5.4}$$

and such an expansion holds for the derivatives of $\rho'_{\Lambda,a,\mu,\varepsilon}$ with respect to a and Λ.

Omitting, for sake of simplicity, the indices $\Lambda, a, \mu, \varepsilon$, we state :

Proposition 5.1 *Assuming that a and Λ satisfy (2.4) (2.5), we have the uniform expansions as ε goes to zero*

$$J_\varepsilon(V) = A + \frac{A}{4}\varepsilon \ln(|\varepsilon|\Lambda) + \frac{1}{2}(C + \frac{A}{6})\varepsilon + \frac{3B\Lambda}{2}\big(\mu^{1/2} + H_\mu(a,a)\big)|\varepsilon| + O(\varepsilon^2(\ln|\varepsilon|)^2)$$

$$\frac{\partial J_\varepsilon}{\partial \Lambda} = \frac{A\varepsilon}{4\Lambda} + \frac{3B}{2}\big(\mu^{1/2} + H_\mu(a,a)\big)|\varepsilon| + O(\varepsilon^2(\ln|\varepsilon|)^2)$$

$$\frac{\partial J_\varepsilon}{\partial a} = \frac{3B\Lambda}{2}\frac{\partial}{\partial a}\big(H_\mu(a,a)\big)|\varepsilon| + O(\varepsilon^2(\ln|\varepsilon|)^2)$$

$$\frac{\partial^2 J_\varepsilon}{\partial \Lambda^2} = -\frac{A\varepsilon}{4\Lambda^2} + O(\varepsilon^2(\ln|\varepsilon|)^2)$$

$$\frac{\partial^2 J_\varepsilon}{\partial \Lambda \partial a} = \frac{3B}{2}\frac{\partial}{\partial a}\big(H_\mu(a,a)\big)|\varepsilon| + O(\varepsilon^2(\ln|\varepsilon|)^2).$$

with

$$A = \int_{\mathbb{R}^3} U_{1,0}^6 = \frac{\pi^2}{4} \qquad B = \int_{\mathbb{R}^3} U_{1,0}^5 = \frac{4\pi}{3} \qquad C = -\frac{1}{2}\int_{\mathbb{R}^3} U_{1,0}^6 \ln U_{1,0} > 0.$$

PROOF. For sake of simplicity, we assume that $\varepsilon > 0$ (the computations are equivalent as $\varepsilon < 0$), and we set $r = |x-a|$. In view of (5.3), we write

$$\int_\Omega (|\nabla V|^2 + \mu V^2) = \int_\Omega (-\Delta V + \mu V)V = \int_\Omega \Big(3U^5 + \mu(U - \frac{(\Lambda\varepsilon)^{\frac{1}{2}}}{r}) + \rho'\Big)V. \tag{5.5}$$

From (5.1)(5.2) we deduce

$$\int_\Omega U^5 V = \int_\Omega U^6 - (\Lambda\varepsilon)^{\frac{1}{2}}\int_\Omega U^5\Big(\frac{1-e^{-\mu^{\frac{1}{2}}r}}{r} + H_\mu(a,x)\Big) + O(\varepsilon^2)$$

noticing that

$$\int_\Omega U^5 = O(\varepsilon^{\frac{1}{2}}). \tag{5.6}$$

One one hand

$$\int_\Omega U^6 = A + O(\varepsilon^3) \quad \text{with} \quad A = \int_{\mathbb{R}^3} U^6 = 4\pi\int_0^\infty \frac{r^2 dr}{(1+r^2)^3} = \frac{\pi^2}{4}.$$

On the other hand, since $d(a,\partial\Omega) \geq \delta > 0$

$$\begin{aligned}
&\int_\Omega U^5\Big(\frac{1-e^{-\mu^{\frac{1}{2}}r}}{r} + H_\mu(a,x)\Big)\\
&\quad= \frac{1}{(\Lambda\varepsilon)^{\frac{1}{2}}}\int_{(\Omega-a)/(\Lambda\varepsilon)} U^5\frac{1-e^{-\mu^{\frac{1}{2}}\Lambda\varepsilon r}}{r}dx + \int_{B(a,R)} U^5 H_\mu(a,x) + O(\varepsilon^{\frac{5}{2}})\\
&\quad= \frac{4\pi}{(\Lambda\varepsilon)^{\frac{1}{2}}}\int_0^{R/(\Lambda\varepsilon)} \frac{1-e^{-\mu^{\frac{1}{2}}\Lambda\varepsilon r}}{(1+r^2)^{\frac{5}{2}}}r dr + H_\mu(a,a)\int_{B(a,R)} U^5 + O\left(\int_{B(a,R)} U^5 r^2 + \varepsilon^{\frac{5}{2}}\right)\\
&\quad= 4\pi B(\Lambda\varepsilon)^{\frac{1}{2}}\big(\mu^{\frac{1}{2}} + H_\mu(a,a)\big) + O(\varepsilon^{\frac{3}{2}})
\end{aligned}$$

with
$$B=\int_{\mathbb{R}^3}U_{1,0}^5=4\pi\int_0^\infty\frac{r^2dr}{(1+r^2)^{\frac52}}=\frac{4\pi}{3}.$$

Concerning the second term in the right hand side of (5.5), denoting by R' the diameter of Ω and using (2.6), we have

$$\begin{aligned}\int_\Omega\mu\Big(U-\frac{(\Lambda\varepsilon)^{\frac12}}{r}\Big)V&=O\left(\int_\Omega\left|U-\frac{(\lambda\varepsilon)^{\frac12}}{r}\right|\right)U\\&=O\left(\varepsilon^2\int_0^{R'/(\Lambda\varepsilon)}\left(\frac1r-\frac{1}{(1+r^2)^{\frac12}}\right)\frac{r^2dr}{(1+r^2)^{\frac12}}\right)\\&=O(\varepsilon^2).\end{aligned}$$

Lastly, noticing that $V=O(U)$ uniformly in Ω and with respect to the parameters a, Λ satisfying (2.4) and (2.5), we have, using (5.4)

$$\begin{aligned}\int_\Omega\rho' V&=O\left(\int_\Omega\Big(\frac{\varepsilon^{\frac12}}{r}(1-e^{-\frac{\varepsilon^2}{r^2}})+\varepsilon^{\frac12}\left(\frac{\varepsilon^2}{r^3}+\frac{\varepsilon^4}{r^5}\right)e^{-\frac{\varepsilon^2}{r^2}}\Big)U+\varepsilon^4\right)\\&=O\left(\varepsilon^2\int_0^{\frac{R'}{\varepsilon}}\left(r(1-e^{-\frac{1}{r^2}})+(\frac1r+\frac{1}{r^2})\right)\frac{dr}{(1+r^2)^{\frac12}}+\varepsilon^4\right)\\&=O(\varepsilon^2)\end{aligned}$$

whence finally

$$\int_\Omega(|\nabla V|^2+\mu V^2)=3A-3B\Lambda\big(\mu^{1/2}+H_\mu(a,a)\big)\varepsilon+O(\varepsilon^2).\tag{5.7}$$

In the same way we have

$$\int_\Omega V_+^6=A-6B\Lambda\big(\mu^{1/2}+H_\mu(a,a)\big)\varepsilon+O(\varepsilon^2).\tag{5.8}$$

Namely, from (5.1)(5.2) and $V=O(U)$ we derive

$$\int_\Omega V_+^6=\int_\Omega U^6-6(\Lambda\varepsilon)^{\frac12}\int_\Omega U^5\Big(\frac{1-e^{-\mu^{\frac12}r}}{r}+H_\mu(a,x)\Big)+O\left(\varepsilon^{\frac32}\int_\Omega U^5+\varepsilon\int_\Omega U^4\right)$$

and the conclusion follows from the previous computations, noticing that

$$\int_\Omega U^4=O(\varepsilon).$$

Then, we write

$$\int_\Omega V_+^{6+\varepsilon}=\int_\Omega V_+^6+\int_\Omega V_+^6(V_+^\varepsilon-1).$$

Noticing that $0\le V_+\le 2\varepsilon^{-\frac12}$

$$V_+^\varepsilon-1=\varepsilon\ln V_++O(\varepsilon^2(\ln\varepsilon)^2)$$

and using the fact that $V_+ = U + O(\varepsilon^{\frac{1}{2}})$ we have

$$V_+^6 = U^6 + O(\varepsilon^{\frac{1}{2}} U^5) \qquad\qquad \ln V_+ = \ln U + O(\frac{\varepsilon^{\frac{1}{2}}}{U})$$

(note that $U \geq \frac{\varepsilon^{\frac{1}{2}}}{R'}$ in Ω) whence

$$V_+^6 \ln V_+ = U^6 \ln U + O\left(\varepsilon^{\frac{1}{2}} U^5 + \varepsilon^{\frac{1}{2}} U^5 |\ln U|\right).$$

We find easily

$$\int_\Omega U^6 \ln U = -\frac{A}{2} \ln(\Lambda\varepsilon) - C + O(\varepsilon^3 |\ln \varepsilon|)$$

and noticing that $\int_\Omega U^5 |\ln U| = O(\varepsilon^{\frac{1}{2}} |\ln \varepsilon|)$, we obtain

$$\int_\Omega V_+^{6+\varepsilon} = \int_\Omega V_+^6 - \frac{A}{2}\varepsilon \ln(|\varepsilon|\Lambda) - C\varepsilon + O(\varepsilon^2 (\ ln|\varepsilon|)^2). \tag{5.9}$$

The first expansion of Proposition 5.1 follows from (5.7), (5.8), (5.9) and the definition (3.23) of J_ε.

The expansions for the derivatives of J_ε are obtained exactly in the same way. □

5.2 Green's function

We study the properties of Green's function $G_\mu(x, y)$ and its regular part $H_\mu(x, y)$. We summarize their properties in the following proposition

Proposition 5.2 *Let $G_\mu(x, y)$ and $H_\mu(x, y)$ be defined in (1.1) and (1.2), respectively. Then we have*

$$H_\mu(x, x) \to -\infty \quad \text{as} \quad d(x, \partial\Omega) \to 0 \tag{5.10}$$

$$|G_\mu(x, y)| \leq \frac{C}{|x - y|} \tag{5.11}$$

$$\mu^{\frac{1}{2}} + \max_{x\in\Omega} H_\mu(x, x) \to -\infty \qquad as \quad \mu \to 0 \tag{5.12}$$

$$\mu^{\frac{1}{2}} + \max_{x\in\Omega} H_\mu(x, x) \to +\infty \qquad as \quad \mu \to +\infty. \tag{5.13}$$

Proof. (5.10) follows from standard argument. Let $x \in \Omega$ be such that $d = d(x, \partial\Omega)$ is small. So there exists a unique point $\bar{x} \in \partial\Omega$ such that $d = |x - \bar{x}|$. Without loss of generality, we may assume $\bar{x} = 0$ and the outer normal at $\bar{x}$ is pointing toward x_N-direction. Let x^* be the reflection point $x^* = (0, ..., 0, -d)$ and consider the following auxiliary function

$$H^*(y, x) = \frac{e^{-\mu^{\frac{1}{2}}|y - x^*|}}{|y - x^*|}$$

Then H^* satisfies $\Delta_y H^* - \mu H^* = 0$ in Ω and on $\partial\Omega$

$$\frac{\partial}{\partial n}(H^*(y, x)) = -\frac{\partial}{\partial n}(\frac{e^{-\mu^{\frac{1}{2}}|y - x|}}{|y - x|}) + O(1)$$

Hence we derive that

$$H(y,x) = -H^*(y,x) + O(1) \tag{5.14}$$

which implies that

$$H(x,x) = -\frac{1}{d(x,\partial\Omega)} + O(1) \tag{5.15}$$

hence (5.10).

From (5.14), we see that as $d(x,\partial\Omega) \to 0$, we have

$$G_\mu(y,x) = \frac{e^{-\mu^{\frac{1}{2}}|y-x|}}{|y-x|} + H^*(y,x) + O(1) \leq \frac{C}{|x-y|} \tag{5.16}$$

On the other hand, if $d(x,\partial\Omega) > d_0 > 0$, then $|H_\mu(y,x)| \leq C$ and (5.11) also holds.

We now prove (5.12). For μ small, we can decompose H_μ as follows:

$$H_\mu(x,y) = c + H_0(x,y) + \hat{H}(x,y) \tag{5.17}$$

where

$$c = \frac{1}{|\Omega|}\int_\Omega H_\mu(x,y) = \frac{1}{\mu|\Omega|}\int_{\partial\Omega}\frac{\partial}{\partial n}\left(\frac{e^{-\mu^{\frac{1}{2}}|y-x|}}{|y-x|}\right) = -\frac{4\pi}{\mu|\Omega|} + O(1) \tag{5.18}$$

and H_0 satisfies

$$-\Delta H_0 = \frac{4\pi}{|\Omega|}, \int_\Omega H_0 = 0, \frac{\partial}{\partial n}H_0 = \frac{\partial}{\partial n}\left(\frac{1}{|y-x|}\right) \text{ on } \partial\Omega$$

and $\hat{H}$ is the remainder term. By simple computations, $\hat{H}$ satisfies

$$\Delta\hat{H} - \mu\hat{H} + O(\mu H_0(x,y)) + O(1) = 0 \text{ in } \Omega, \int_\Omega \hat{H} = 0, \quad \frac{\partial}{\partial n}\hat{H} = O(1) \text{ on } \partial\Omega$$

which shows that $\hat{H} = O(1)$. Thus

$$\mu^{\frac{1}{2}} + \max_{x\in\Omega} H_\mu(x,x) \leq -\frac{4\pi}{\mu|\Omega|} + O(1) \to -\infty$$

as $\mu \to 0$. (5.12) is thus proved.

To prove (5.13), we choose a point $x_0 \in \Omega$ such that $d(x_0,\partial\Omega) = \max_{x\in\Omega} d(x,\partial\Omega)$. Then, since $\frac{\partial}{\partial n}\left(\frac{e^{-\mu^{\frac{1}{2}}|x_0-x|}}{|x_0-x|}\right) = O(e^{-\frac{\mu^{\frac{1}{2}}}{2}d(x_0,\partial\Omega)})$ on $\partial\Omega$, for μ large enough we see that

$$\mu^{\frac{1}{2}} + \max_{x\in\Omega} H_\mu(x,x) \geq \mu^{\frac{1}{2}} + H(x_0,x_0) \geq \mu^{\frac{1}{2}} + O(e^{-\frac{\mu^{\frac{1}{2}}}{2}d(x_0,\partial\Omega)}) \to +\infty$$

as $\mu \to +\infty$, which proves (5.13). $\square$

References

[1] ADIMURTHI, G. MANCINI, *The Neumann problem for elliptic equations with critical nonlinearity, "A tribute in honour of G. Prodi"*, Scuola Norm. Sup. Pisa (1991) 9-25.

[2] ADIMURTHI, G. MANCINI, *Geometry and topology of the boundary in the critical Neumann problem*, J. reine angew. Math. **456** (1994) 1-18.

[3] ADIMURTHI, F. PACELLA, S.L. YADAVA, *Interaction between the geometry of the boundary and positive solutions of a semilinear Neumann problem with critical nonlinearity*, J. Funct. Anal. **113** (1993) 318-350.

[4] A. BAHRI, *Critical points at infinity in some variational problems*, Pitman Research Notes in Math. Series **182**, Longman (1989).

[5] P. BATES, G. FUSCO, *Equilibria with many nuclei for the Cahn-Hilliard equation*, J. Diff. Equ. **160** (2000) 283-356.

[6] L. CAFFARELLI, B. GIDAS, J. SPRUCK, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, in Comm. Pure Appl. Math. **42** (1989) 271-297.

[7] G. CERAMI, J. WEI, *Multiplicity of multiple interior peaks solutions for some singularly perturbed Neumann problems*, Intern. Math. Res. Notes **12** (1998) 601-626.

[8] E.N. DANCER, S. YAN, *Multipeak solutions for a singularly perturbed Neumann problem*, Pacific J. Math. **189** (1999) 241-262.

[9] M. DEL PINO, P. FELMER, M. MUSSO, *Two-bubble solutions in the super-critical Bahri-Coron's problem*, to appear in Calc. Var. and Part. Diff. Eq.

[10] N. GHOUSSOUB, C. GUI, M. ZHU, *On a singularly perturbed Neumann problem with the critical exponent*. Comm. Partial Differential Equations **26** (2001) 1929-1946.

[11] M. GROSSI, A. PISTOIA, *On the effect of critical points of distance function in superlinear elliptic problems*, Adv. Diff. Equ. **5** (2000) 1397-1420.

[12] M. GROSSI, A. PISTOIA, J. WEI, *Existence of multipeak solutions for a semilinear elliptic problem via nonsmooth critical point theory*, Calc. Var. PDE **11** (2000) 143-175.

[13] C. GUI, *Multi-peak solutions for a semilinear Neumann problem*, Duke Math. J. **84** (1996) 739-769.

[14] A. GIERER, H. MEINHARDT, *A theory of biological pattern formation,* Kybernetik (Berlin) **12** (1972) 30-39.

[15] C. GUI, C.S. LIN, *Estimates for boundary-bubbling solutions to an elliptic Neumann problem*, J. Reine Angew. Math. **546** (2002) 201–235.

[16] C. GUI, J. WEI, *Multiple interior peak solutions for some singularly perturbed Neumann problems*, J. Diff. Equ. **158** (1999) 1-27.

[17] C. GUI, J. WEI, *On the existence of arbitrary number of bubbles for some semilinear elliptic equations with critical Sobolev exponent*, to appear.

[18] C. Gui, J. Wei, *On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems*, Canad. J. Math. **52** (2000) 522-538.

[19] C. Gui, J. Wei, M. Winter, *Multiple boundary peak solutions for some singularly perturbed Neumann problems*, Ann. Inst. H. Poincar, Anal. non-linaire **17** (2000) 47-82.

[20] S. Khenissy, O. Rey, *A criterion for existence of solutions to the supercritical Bahri-Coron's problem*, to appear in Houston J. Math.

[21] M. Kowalczyk, *Multiple spike layers in the shadow Gierer-Meinhardt system : existence of equilibria and quasi-invariant manifold*, Duke Math. J. **98** (1999) 59-111.

[22] Y.Y. Li, *On a singularly perturbed equation with Neumann boundary condition*, Comm. Part. Diff. Equ. **23** (1998) 487-545.

[23] C.S. Lin, W.M. Ni, *On the Diffusion Coefficient of a Semilinear Neumann Problem*, Springer Lecture Notes **1340**, Springer, New York, Berlin (1986).

[24] C.S. Lin, W.N. Ni, I. Takagi, *Large amplitude stationary solutions to a chemotaxis system*, J. Diff Equ. **72** (1988) 1-27.

[25] S. Maier-Paape, K. Schmitt, Z.Q. Wang, *On Neumann problems for semilinear elliptic equations with critical nonlinearity : existence and symmetry of multi-peaked solutions*, Comm. Part. Diff. Equ. **22** (1997), 1493-1527.

[26] W.-M. Ni, *Diffusion, cross-diffusion, and their spike-layer steady states*, Notices of Amer. Math. soc. **45** (1998) 9-18.

[27] W.N. Ni, X.B. Pan, I. Takagi, *Singular behavior of least-energy solutions of a semilinear Neumann problem involving critical Sobolev exponents*, Duke Math. J. **67** (1992) 1-20.

[28] W.N. Ni, I. Takagi, *On the shape of least-energy solutions to a semi-linear problem Neumann problem*, Comm. Pure Appl. Math. **44** (1991) 819-851.

[29] W.M. Ni, I. Takagi, *Locating the peaks of least-energy solutions to a semi-linear Neumann problem*, Duke Math. J. **70** (1993) 247-281.

[30] O. Rey, *The role of the Green's function in a nonlinear elliptic problem involving the critical Sobolev exponent*, J. Funct. Anal. **89** (1990) 1-52.

[31] O. Rey, *An elliptic Neumann problem with critical nonlinearity in three dimensional domains*, Comm. Contemp. Math. **1** (1999) 405-449.

[32] O. Rey, *The question of interior blow-up points for an elliptic Neumann problem : the critical case*, J. Math. Pures Appl. **81** (2002) 655-696.

[33] J. Wei, M. Winter, *Stationary solutions for the Cahn-Hilliard equation*, Ann. Inst. H. Poincaré, Anal. non lin. **15** (1998) 459-482.

[34] X.J. Wang, *Neumann problem of semilinear elliptic equations involving critical Sobolev exponents*, J. Diff. Equ. **93** (1991) 283-310.

[35] Z.Q. Wang, *The effect of domain geometry on the number of positive solutions of Neumann problems with critical exponents*, Diff. and Integ. Equ. **8** (1995) 1533-1554.

[36] Z.Q. Wang, *High energy and multi-peaked solutions for a nonlinear Neumann problem with critical exponent*, Proc. Roy. Soc. Edimburgh **125 A** (1995) 1003-1029.

[37] Z.Q. Wang, *Construction of multi-peaked solution for a nonlinear Neumann problem with critical exponent*, J. Nonlinear Anal. TMA **27** (1996), 1281-1306.

[38] J. Wei, *On the interior spike layer solutions of singularly perturbed semilinear Neumann problems*, Tohoku Math. J. **50** (1998) 159-178.

[39] J. Wei, X. Xu, *Uniqueness and a priori estimates for some nonlinear elliptic Neumann equations in* $\mathbb{R}^3$. Pacific J. Math., to appear.

[40] S. Yan, *On the number of interior multipeak solutions for singularly perturbed Neumann problems*, Top. Meth. Nonlin. Anal. **12** (1998) 61-78.

[41] M. Zhu, *Uniqueness results through a priori estimates, I. A three dimensional Neumann problem*, J. Diff. Equ. **154** (1999) 284-317.